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CONVERGENCE OF ITERATIVE NEWTON'S METHOD FOR THE SOLUTION NUMERICAL EQUATIONS

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Abstract:In this paper an Iterative Newton's method for the solution of simple and multiple roots of an equation f(x)=0 is given and it is proved that this method has second order convergence.

Key words: N-R Method, Iterative Method.

1. INTRODUCTION

Let us consider the equation,

$$f(x) = 0 \qquad (1.1)$$

where f may be algebraic, transcendental or combined of both.

If η be a simple root of equation (1.1), the Newton – Raphson method for finding η is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n=0,1,2,\ldots) \qquad (1.2)$$

which has quadratic convergence, and this method converges as long as

$$\rho_\eta = \frac{ff''}{f'^2}$$

(1.3)

for all x and for each n

and if ηbe root of (1.1) with multiplicity m, then the Newton Raphson method for multiple roots is defined as

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \ n=(0,1,2,\ldots.) \qquad (1.4)$$

As noted by Jain.et.al [1] when equation (1.1) has a multiple root, most of the methods exist for solving a simple root of $f(x) = 0$ have only linear rate of convergence.

In this paper, we develop an Iterative Newton's method in section (2) and also it is shown that the method is same for solving equation (1.1) whether it has a simple root or a multiple root in the same section. Whereas, in section (3) we consider some numerical examples as given in S.S. Sastry [2] and M.K.Jain [1] and compare this method with the methods (1.2) and (1.4).

2.ITERATIVE NEWTON'S METHOD

By introducing a computational parameter $\alpha_n > 0$ and extrapolating the method (1.4), we obtain

$$x_{n+1} = (1-\bar{\alpha}_n)x_n + \bar{\alpha}_n\left[x_n - m\frac{f(x_n)}{f'(x_n)}\right]$$

or

$$x_{n+1} = x_n - \bar{\alpha}_n m\frac{f(x_n)}{f'(x_n)} \quad (n = 0,1,2,\ldots), \qquad (2.1)$$

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where m is the multiplicity of root of (1.1)

Since $\bar{\alpha}_n$ and m being two scalar quantities, we let $\bar{\alpha}_n.m = \alpha_n$ (say)

Now the method takes the form

$$x_{n+1} = x_n - \alpha_n \frac{f(x_n)}{f'(x_n)} \quad , \text{(n=0,1,2,…)} \tag{2.2}$$

It is well known that the iterative method

$$x_{n+1} = \phi(x_n) \tag{2.3}$$

converges if $|\phi'(x)| < 1$ for all *x* and for each *n* (2.4)

from (2.4), we obtain the condition for convergence of the method (2.2)as

$$\mu = |1 - \alpha_n + \alpha_n \rho_n| < 1 \tag{2.5}$$

for all *x* and for each *n*

where

$$\rho_n = \frac{f(x_n) f''(x_n)}{f'^2(x_n)}$$

we need to find a real value of α_n for each iteration, which minimizes μ of (2.5)

It can be easily verified that the condition (2.5) holds true if

$$0 < \alpha_n < \frac{2}{1-\rho_n} \quad , \quad (n = 0,1,2,\ldots) \tag{2.6}$$

Taking the mean of the bounds of α_n and fixing it as choice for α_n, we obtain

$$\alpha_n = \frac{1}{1-\rho_n} \quad , \quad (n = 0,1,2,\ldots) \tag{2.7}$$

since the choice for α_n of (2.7) agrees with the condition (2.6), the condition (2.5) automatically satisfies and hence the method (2.2) converges.

It is to note that the method (1.4) is generally applied for finding a multiple root of equation (1.1) knowing the multiplicity of root in advance. It is further interesting to note that the method (2.2) can be applied to find a simple or multiple root of equation (1.1), irrespective of the value of multiplicity *m≥ 1* as long as

$$\bar{\alpha}_n.m = \alpha_n = \frac{1}{1-\rho_n} \quad \text{for all } x \text{ and for each } n$$

3. ITERATIVE NEWTON'S METHOD HAS A SECOND ORDER CONVERGENCE

Let η be the root of the equation (1.1) and e_{n+1}, e_n be errors when x_{n+1}, x_n are the approximate values of the root.

Then, we have

$$x_{n+1} = \eta + e_{n+1} \tag{3.1}$$

$$x_n = \eta + e_n$$

Substituting these values x_{n+1} and x_n in (2.2), we have

$$x_{n+1} = x_n - \frac{1}{1 - \frac{ff''}{f'^2}} \frac{f}{f'}$$

$$= x_n - \frac{ff'}{f'^2 - ff''}$$

Putting (3.1) in the above method, we obtain

$$e_{n+1} = e_n - \frac{\left\{\left[f + e_n f' + \frac{e_n^2}{2} f'' + \frac{e_n^3}{6} f''' + \dots\right].\left[f' + e_n f'' + \frac{e_n^2}{2} f''' + \frac{e_n^3}{6} f^{iv} + \dots.\right]\right\}}{\left\{\left[f' + e_n f'' + \frac{e_n^2}{2} f''' + \frac{e_n^3}{6} f^{iv} + \dots\right]^2 - \left[f + e_n f' + \frac{e_n^2}{2} f'' + \frac{e_n^3}{6} f''' \dots\right].\left[f'' + e_n f''' + \frac{e_n^2}{2} f^{iv} + \frac{e_n^3}{6} f^{v} + \dots.\right]\right\}}$$

$$= e_n - \frac{\left\{\begin{bmatrix} ff' + e_n f'^2 + \frac{e_n^2}{2} f' f'' + \frac{e_n^3}{6} f' f''' + e_n ff'' + e_n^2 f' f'' + \\ \frac{e_n^3}{2} f''^2 + \frac{e_n^4}{6} f'' f''' + \frac{e_n^2}{2} ff''' + \frac{e_n^3}{2} f' f''' + \frac{e_n^4}{4} f'' f''' + \\ \frac{e_n^5}{12} f'''^2 + \frac{e_n^3}{6} ff^{iv} + \frac{e_n^4}{6} f' f^{iv} + \frac{e_n^5}{12} f'' f^{iv} + \frac{e_n^6}{36} f''' f^{iv} \dots \end{bmatrix}\right\}}{\left\{\begin{array}{l} \left[f'^2 + e_n^2 f''^2 + \frac{e_n^4}{4} f'''^2 + 2e_n f' f'' + e_n^3 f'' f''' + e_n^2 f' f'''\right] - \\ \begin{bmatrix} ff'' + e_n ff''' + \frac{e_n^2}{2} ff^{(iv)} + \frac{e_n^3}{6} ff^{(v)} + e_n f' f'' + e_n^2 ff'' + \frac{e_n^3}{2} f' f^{(iv)} + \\ \frac{e_n^4}{6} f' f^{v} + \frac{e_n^2}{2} f''^2 + \frac{e_n^3}{2} f'' f''' + \frac{e_n^4}{4} f'' f^{iv} + \frac{e_n^3}{6} f'' f''' + \frac{e_n^4}{6} f'''^2 + .. \end{bmatrix} \end{array}\right\}}$$

$$= e_n - \frac{\left\{\begin{bmatrix} ff' + e_n f'^2 + \frac{e_n^2}{2} f' f'' + \frac{e_n^3}{6} f' f''' + e_n ff'' + e_n^2 f' f'' + \\ \frac{e_n^3}{2} f''^2 + \frac{e_n^4}{6} f'' f''' + \frac{e_n^2}{2} ff''' + \frac{e_n^3}{2} f' f''' + \frac{e_n^4}{4} f'' f''' + \\ \frac{e_n^5}{12} f'''^2 + \frac{e_n^3}{6} ff^{iv} + \frac{e_n^4}{6} f' f^{iv} + \frac{e_n^5}{12} f'' f^{iv} + \frac{e_n^6}{36} f''' f^{iv} \dots \end{bmatrix}\right\}}{\left\{\begin{array}{l} \left[f'^2 + e_n^2 f''^2 + \frac{e_n^4}{4} f'''^2 + 2e_n f' f'' + e_n^3 f'' f''' + e_n^2 f' f'''\right] - \\ \begin{bmatrix} ff'' + e_n ff''' + \frac{e_n^2}{2} ff^{(iv)} + \frac{e_n^3}{6} ff^{(v)} + e_n f' f'' + e_n^2 ff'' + \frac{e_n^3}{2} f' f^{(iv)} + \\ \frac{e_n^4}{6} f' f^{v} + \frac{e_n^2}{2} f''^2 + \frac{e_n^3}{2} f'' f''' + \frac{e_n^4}{4} f'' f^{iv} + \frac{e_n^3}{6} f'' f''' + \frac{e_n^4}{6} f'''^2 + .. \end{bmatrix} \end{array}\right\}}$$

$$= e_n - \frac{[e_n(f'^2) + e_n^2(\frac{f'f''}{2} + f'f'') + e_n^3(\frac{f'f'''}{2} + \frac{f''^2}{2} + \frac{f'f'''}{6}) + ...]}{f'^2 + e_n(2f'f'' - f'f'') + e_n^2(f''^2 + f'f''' - f'f''' - \frac{f''^2}{2}) + ...}$$

$$e_{n+1} = e_n - \frac{\left\{\left[e_n f'^2 + \frac{3}{2}e_n^2 f'f'' + \frac{e_n^3}{6}\left(3f''^2 + 4f'f'''\right) + ...\right]\right\}}{\left\{f'^2 + e_n f'f'' + \frac{e_n^2}{2}f''^2 + ...\right\}}$$

$$= \frac{1}{f'^2}\left[e_n f'^2 + e_n^2 f'f'' + \frac{e_n^3}{2}f''^2 + ...\right] - \frac{1}{f'^2}$$

$$\left[e_n f'^2 + \frac{3}{2}e_n^2 f'f'' + \frac{e_n^3}{6}\left(3f''^2 + 4f'f'''\right) + ..\right]$$

$$= e_n^2\left[f'f'' - \frac{3}{2}f'f''\right] / f'^2$$

$$e_{n+1} \approx e^2{}_n.k \tag{3.2}$$

where $K = -\frac{1}{2}\left(\frac{f''}{f'}\right)$

This shows that the subsequent error at each step, is proportional to the square of the previous error and as such the convergence is quadratic**.**

4.NUMERIACAL EXAMPLES

We consider some examples given in Sastry [2] and Jain [1] for finding the simple and multiple roots of an equation using the methods (1.2),(1.4)and(2.2)discussed in this paper and the successive approximations of the roots are tabulated below until the functional value becomes negligible.

Table 4.1

Finding the double root of ***f(x) = x^3- x^2-x+1* with *$x_1 = 4$***.

No	**Method (1.4)**	**Method(2.2)**
n	x_{n+1}	X_{n+1}
1	1.692307692	0.694915254
2	1.078870497	0.968116511
3	1.001468285	0.999737583
4	1.0000005	1.000001101

Table 4.2

Finding the triple root of ***f(x) = $27x^5+27x^4+36x^3+28x^2$+9x+1* lies in (-1, 0) with *x_0*= -1**

No	**Method (1.4)**	**Method(2.2)**
n	x_{n+1}	X_{n+1}
0	-0.454545456	-0.185185186
1	-0.336914194	-0.332661525
2	-0.333337084	-0.333313205

5. CONCLUSION

Newton's Iterative method for the solution of simple and multiple roots of an equation is developed and convergence of this method is discussed. It is shown that the method is the same for simple and multiple roots of an equation $f(x)=0.$ The advantage of this method is, the equations having multiple roots can be solved without knowing the multiplicity in advance.

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